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ON SHEAR VIBRATIONS OF A PUNCH ON THE SURFACE OF A PRE-STRESSED HALF-SPACE†

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Using a generalization [1, 2] of the method of factorization for integral equations, the kernels of which can have branch points on the real axis, a method of investigating the dynamics of a rigid punch which performs shear vibrations on the surface of a pre-stressed half-space is developed. The structure of the solution enables an effective analysis to be made of the influence of the initial stresses on the wave process both under the punch, and outside of it. The vibrations at the edges of the punch, due to the effect of shear waves in the pre-stressed medium, is represented explicitly.

PREVIOUSLY [3–5], using the linearized theory of the propagation of elastic waves [6, 7], a method of studying the influence of initial stresses on the wave field under a punch and on the free surface of the medium, was developed. The method was based on reducing the boundary value problem of the vibrations of the punch to a first-order integral equation, the solution of which was constructed by the factorization method [8]. However, this method did not allow for the presence in the kernel of the integral equation of branch points on the real axis, which is characteristic of contact problems for a half-space. For low-frequency vibrations of the punch, the influence of branch points can be ignored [3–5, 8], but in the case of higher frequencies [9] branch points must be taken into account.

1. THE BOUNDARY-VALUE PROBLEM OF THE EXCITATION OF A PRE-STRESSED MEDIUM

We introduce a Lagrangian system of coordinates x_1, x_2, x_3 associated with the natural (undeformed) state of a body occupying the region $|x_1|$, $|x_2|\leqslant\infty$, $x_3\leqslant 0$. When investigating processes in a pre-stressed body, three states (configurations) are distinguished [6, 7, 10]: the natural (unstressed) state, the initially deformed state and the perturbed state at a given instant of time.

The boundary-value problem of the excitation of a pre-stressed medium by an oscillating load distributed in the region Ω is described by linearized equations of motion with boundary conditions [6, 7, 10]

$$\nabla\cdot\underline{\underline{\Theta}} = \rho\partial^2\underline{\mathbf{u}}/\partial t^2, \quad \underline{\underline{\Theta}} = \underline{\underline{\Sigma}}^*\cdot\underline{\underline{C}}^\circ + \underline{\underline{\Sigma}}^\circ\cdot\nabla\underline{\mathbf{u}}$$
$$\underline{\mathbf{n}}\cdot\underline{\underline{\Theta}} = \begin{cases} \underline{\mathbf{f}}(x_1, x_2), & x_3 = 0,\ x_1, x_2 \in \Omega \\ 0, & x_3 = 0,\ x_1, x_2 \notin \Omega \end{cases} \tag{1.1}$$

Here ∇ is the Hamilton operator, defined in the coordinates of the natural configuration, **n** is the external normal to the surface $x_3 = 0$, **f** is the given stress vector, **u** is the displacement vector of an arbitrary point of the medium in the perturbed configuration $\underline{\underline{C}}^\circ$ is the deformation gradient of the location characterizing the initially deformed state in this case, Σ^* is the generalized stress tensor

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[6, 7] (of Lagrange [7]) and Kirchhoff [10]), which defines the law of state in a body with initial stresses, $\underline{\underline{\Sigma}}^\circ$ is the initial stress tensor [7, 10]

$$\underline{\underline{\Sigma}}^* = \frac{1}{2}\frac{\partial^2 W^\circ}{\partial \underline{\underline{E}}^{\circ 2}} \cdot\cdot [\nabla \underline{u}\cdot \underline{\underline{C}}^{\circ T} + \underline{\underline{C}}^\circ \cdot \nabla \underline{u}^T], \qquad \underline{\underline{\Sigma}}^\circ = \frac{\partial W^\circ}{\partial \underline{\underline{E}}^\circ} \tag{1.2}$$

$$\overset{\circ}{E}_{mn} = {}^1/_2 [\overset{\circ}{u}_{m,n} + \overset{\circ}{u}_{n,m} + \overset{\circ}{u}_{i,m}\overset{\circ}{u}_{i,n}]$$

where $\underline{\underline{E}}^\circ$ is the Cauchy–Green strain tensor, the index $^\circ$ denotes values relating to the initially deformed state, differentiation with respect to Lagrangian coordinates is denoted by the subscript after the comma and W° is the elastic potential which, in this case, is assumed to be a tice continuously differentiable function. Below, we shall consider media which have an elastic potential of the form [6, 7]

$$W = W(B_1, B_2, B_3, B_4, B_5) \tag{1.3}$$

$B_1 = E_{nn}$, $B_2 = E_{nm}E_{mn}$, $B_3 = E_{nm}E_{mi}E_{in}$; $B_4 = E_{33}$, $B_5 = E_{3n}E_{n3}$, $n = 1, 2$ where B_1, B_2 and B_3 are algebraic invariants of the strain tensor.

2. SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF SHEAR VIBRATIONS OF A PRE-STRESSED HYPERELASTIC MEDIUM

Relations (1.1) and (1.2) describe the boundary-value problem of the excitation of an initially deformed medium in the most general case, irrespective of the form of the initial stressed state, the type of surface load or form of the elastic potential.

In the presence of a uniform initial deformation of the form

$$u_n^\circ = \delta_{in}(\lambda_i - 1)x_i, \quad \lambda_i = \text{const}, \quad i = 1, 2, 3$$

(1.1) and (1.2) become much simpler [6, 7]. In that case the boundary-value problem of excitation can be written in the form [6, 7] (δ_{in} is the Kronecker delta)

$$A_2 u_{2,11} + A_1 u_{2,33} - \rho \ddot{u}_2 = 0$$

$$A_1 u_{2,3} = \begin{cases} q(x_1), & x_1 \in [-a, a] \\ 0, & x_1 \notin [-a, a] \end{cases} \tag{2.1}$$

$$A_1 = \lambda_2^2 \mu_{32} + \sigma_{33}^{*\circ}, \quad A_2 = \lambda_2^2 \mu_{12} + \sigma_{11}^{*\circ}$$

$$\mu_{in} = \left[\frac{\partial}{\partial B_2^\circ} + {}^3/_4 (r_i + r_n)\frac{\partial}{\partial B_3^\circ} + {}^1/_2(\delta_{i1}\delta_{n3} + \delta_{i3}\delta_{n1} + \right.$$
$$\left. + \delta_{i2}\delta_{n3} + \delta_{i3}\delta_{n2})\frac{\partial}{\partial B_4^\circ}\right] W^\circ, \quad r_n = \lambda_n^2 - 1 \tag{2.2}$$

$$\sigma_{in}^{*\circ} = \delta_{in}\left[\frac{\partial}{\partial B_1^\circ} + r_n \frac{\partial}{\partial B_2^\circ} + {}^3/_4 r_n^2 \frac{\partial}{\partial B_3^\circ} + \delta_{n3}\frac{\partial}{\partial B_4^\circ}\right] W^\circ$$

Using the methods of operational calculus and the limiting absorption principle, the solution of boundary-value problem (2.1) can be written in dimensionless form

$$u_2'(x_1', x_3') = (2\pi)^{-1}\int_{-1}^{1} k(x_1' - \xi', x_3')\, q'(\xi')\, d\xi' \tag{2.3}$$

$$k(s', x_3') = \int_\Gamma K(\alpha, x_3') \exp(i\alpha s')\, d\alpha, \qquad K(\alpha, t) = \frac{c \exp(\sqrt{\alpha^2 - \varkappa^2}\, t)}{\sqrt{\alpha^2 - \varkappa^2}}$$

$$\varkappa = \sqrt{\frac{\rho}{A_2}}\,\omega a, \quad c = \frac{1}{\sqrt{A_1A_2}}, \quad x_i' = \frac{x_i}{a}, \quad \xi' = \frac{\xi}{a}$$

$$s' = \frac{s}{a}, \quad u' = \frac{u}{a}, \quad q' = \frac{q}{\mu}, \quad A_i' = \frac{A_i}{\mu}$$

Primes will be omitted below. The contour Γ coincides with the real axis, only deviating from it when going upwards round the point $-\varkappa$ or downwards round $\varkappa$. This ensures the uniqueness of the solution of the problem [8].

3. INTEGRAL EQUATION OF THE PROBLEM OF THE SHEAR VIBRATIONS OF A PUNCH

Expressions (2.3) define the displacements of an arbitrary point of the medium under the action of a load $\mathbf{q}(\xi)$ which is defined in section $[-1, 1]$. For the problem of the shear vibrations of a punch on the surface of a half-space, these expressions can be rewritten in the form

$$\mathbf{u}_2(x_1, 0) = (2\pi)^{-1}\int_{-1}^{1} k_0(x_1-\xi)\,\mathbf{q}(\xi)\,d\xi, \quad |x_1| \leqslant 1 \tag{3.1}$$

$$k_0(s) = \int_\Gamma K_0(\alpha)\exp(i\alpha s)\,d\alpha, \quad K_0(\alpha) = \frac{c}{\sqrt{\alpha^2-\varkappa^2}} \tag{3.2}$$

Expression (3.1) is a first-order integral equation for the unknown distribution function of contact stresses $\mathbf{q}(\xi)$. It is clear that the function $K_0(\alpha)$is analytic in the complex plane with two cuts which do not intersect one another, drawn from the branch points $\pm\varkappa$ strictly in the first and third quadrants. As $\alpha\to\infty$ we have $|\alpha|K_0(\alpha)\approx c+O(\alpha^{-2})$.

In view of the properties of $K_0(\alpha)$ described above, a number of numerical and asymptotic methods can be used to solve the integral equation (3.1) which enable an approximate solution of the problem to be constructed (see [9, 11, 12], for example). Following the approach described in [1, 2], we write the function $k_0(\alpha)$ in the form

$$K_0(\alpha) = K_+K_-, \quad K_\pm(\alpha) = \frac{c_\pm}{\sqrt{\alpha\mp\varkappa}}, \quad c_\pm = \sqrt{c}\exp\left(\pm i\frac{\pi}{4}\right) \tag{3.3}$$

The constants c_+ and c_- are chosen to satisfy the condition

$$K_+(-\alpha) = K_-(\alpha) \tag{3.4}$$

Taking into account the properties of the function $K_0(\alpha)$ noted above, after some manipulation we reduce (3.1) to a system of second-order integral equations of the following form (the upper and then the lower signs are taken in succession)

$$\mathbf{X}(z,\pm) = -\frac{1}{2\pi i}\int_{\Gamma_1}\mathbf{X}(\alpha,\pm)\,\mathbf{P}(\alpha,z)\,d\alpha + \boldsymbol{\alpha}(z,\pm), \quad \operatorname{Im} z \leqslant 0 \tag{3.5}$$

$$\mathbf{P}(\alpha, z) = K_-(\alpha)e^{-2i\alpha}[K_+(\alpha)(\alpha+z)]^{-1}$$

$$\boldsymbol{\alpha}(z,\pm) = -\frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{F}(\alpha)\pm\mathbf{F}(-\alpha)}{K_+(\alpha)}e^{i\alpha}\frac{d\alpha}{\alpha+z}$$

$$\mathbf{X}(z,\pm) = [\Phi^+(z)\pm\Phi^-(z)]e^{iz}[K_-(z)]^{-1}$$

for the auxiliary unknowns $\mathbf{X}(z,\pm)$, which are combinations of $\Phi^+(z)$ and $\Phi^-(z)$, the Fourier transforms of functions which are continuations of the right-hand side of Eq. (3.1) to the region $x>1\,(\varphi^+(x))$ and $x<-1\,(\varphi^- x))$, respectively, and $\mathbf{F}(\alpha)$ is the Fourier transform of the function $\mathbf{f}(x)$.

The contour Γ_1 lies strictly above Γ, but does not leave the regularity band, which is a certain neighbourhood of the contour Γ. In this case the solution of the integral equation (3.1) is given by the formula [1, 2, 8]

$$\mathbf{q}(\mathbf{x}) = \frac{1}{2\pi}\int_{\Gamma} [\mathbf{F}(\mathbf{u}) + \Phi^+(\mathbf{u}) + \Phi^-(\mathbf{u})]\, e^{-i\mathbf{u}\mathbf{x}} [K(\mathbf{u})]^{-1} d\mathbf{u}, \qquad \mathbf{x} \leqslant 1 \tag{3.6}$$

$$\Phi^\pm(\mathbf{u}) = {}^1/_2[\mathbf{X}(\mp\mathbf{u}, +) \pm \mathbf{X}(\mp\mathbf{u}, -)] K_\pm(\mathbf{u}) e^{\pm i\mathbf{u}}$$

The behaviour of the free surface outside the punch is defined by the expression

$$\varphi^\pm(\mathbf{x}) = \frac{1}{4\pi}\int_{\Gamma} \Phi^\pm(\mathbf{u})\, e^{-i\mathbf{u}\mathbf{x}} d\mathbf{u}, \qquad \pm\mathbf{x} > 1 \tag{3.7}$$

To illustrate the solution of system (3.5), let us deform the contour Γ_1 in the lower half-plane (in the region of regularity of the functions $K_-(\alpha)$, $\mathbf{X}(\alpha, \pm)$), so that it goes around the cut from the branch point $-\varkappa$ to an infinite point parallel to the imaginary axis (from $\varkappa - i^\infty$ to $-\varkappa$ to the left of the cut and from $-\varkappa$ to $-\varkappa - i^\infty$ to the right of the cut). Representing the integrals for the left- and right-hand sides of the cut in the form of a sum and taking account of the relation between the values of $K_+(\alpha)$ on those sides, we can write (3.5) in the form

$$\mathbf{X}(z, \pm) = \pm\sum_{k=1}^{N} \mathbf{X}(\mathbf{z}_k, \pm)\frac{\mathbf{r}_k}{\mathbf{z} - \mathbf{z}_k} + \alpha(\mathbf{z}, \pm) + O(\exp(2i\mathbf{z}_N)), \qquad \operatorname{Im}\mathbf{z} \leqslant 0$$

$$\mathbf{r}_k = \frac{1}{\pi i}\frac{K_-(-\mathbf{z}_k)}{K_+(-\mathbf{z}_k)}\exp(2i\mathbf{z}_k)\Delta\mathbf{z}_k, \qquad \Delta\mathbf{z}_k = \mathbf{z}_{k-1} - \mathbf{z}_k \tag{3.8}$$

where $-\mathbf{z}_k = -\varkappa - it_k$ ($k = 1, 2, \ldots, \mathrm{N}$) are points situated on the sides of the cut $[-\varkappa, -\varkappa - i^\infty]$. In this way (3.5) has been reduced to the finite system of algebraic equations (3.8), the solutions of which can be written in the form:

$$\mathbf{X}(\mathbf{z}, \pm) = \pm\sum_{k=1}^{N}\frac{\mathbf{r}_k}{\mathbf{z} - \mathbf{z}_k}\left[\sum_{l=1}^{N} B_{kl}{}^{\pm}\alpha(\mathbf{z}_l, \pm)\right] + \alpha(\mathbf{z}, \pm) \tag{3.9}$$

where B_{kl} are the elements of the matrix inverse to

$$A = (\delta_{lk} - \mathbf{r}_k/(\mathbf{z}_l + \mathbf{z}_k))$$

With the help of formulas (3.6) and (3.7), the functions $\mathbf{q}(x)$ and $\varphi^+(\mathbf{x})$, $\varphi^-(\mathbf{x})$ are easily found. Let us consider the case

$$\mathbf{f}(\mathbf{x}) = e^{i\eta\mathbf{x}}, \qquad |\mathbf{x}| \leqslant 1 \tag{3.10}$$

Following [8], we continue this function to the intervals $\mathbf{x} > 1$ and $\mathbf{x} < -1$. This gives us the new functions

$$\varphi_1{}^+(\mathbf{x}) = \varphi^+(\mathbf{x}) - e^{i\eta\mathbf{x}}, \ \mathbf{x} > 1, \quad \varphi_1{}^-(\mathbf{x}) = \varphi^-(\mathbf{x}) - e^{i\eta\mathbf{x}}, \ \mathbf{x} < -1 \tag{3.11}$$

Then the functions $\alpha(\mathbf{z}, \pm)$ on the right-hand sides of (3.5) are represented in the form

$$\alpha(\mathbf{z}, \pm) = i\left[\frac{e^{i\eta}}{K_-(\eta)(\mathbf{z} - \eta)} \pm \frac{e^{-i\eta}}{K_+(\eta)(\mathbf{z} + \eta)}\right] \tag{3.12}$$

Using (3.9) and (3.12), and applying the formulas of the operational calculus [13] to (3.6) and (3.7), we obtain

$$\mathbf{q}(\mathbf{x}) = -\frac{e^{i\eta x}}{K_0(\eta)} + S^+(\eta, 1 - \mathbf{x}) + S^-(\eta, 1 + \mathbf{x}), \qquad |\mathbf{x}| \leqslant 1 \tag{3.13}$$

$$S^\pm(\eta, t) = \frac{e^{\pm i\eta(1-t)}}{K_0(\eta)}\operatorname{erf}\sqrt{-i(\varkappa \pm \eta)\mathbf{t}} + \frac{e^{i\varkappa\mathbf{t}}}{\sqrt{\pi\mathbf{t}}}\mathbf{R}^\pm + \frac{i}{2}\sum_{k=1}^{N}\frac{e^{-i\mathbf{z}_k\mathbf{t}}}{K_+(\mathbf{z}_k)}\mathbf{P}_k{}^\pm(t) \tag{3.14}$$

$$\varphi^{\pm}(x) = e^{i\eta x}[1 - \operatorname{erf}\sqrt{-i(\varkappa - \eta)(x \mp 1)}] + \\ + \frac{i}{2}\sum_{k=1}^{N} K_-(z_k)\, e^{iz_k(x\mp 1)} S_k^{\pm}(x \mp 1), \quad \pm x > 1 \tag{3.15}$$

$$R^{\pm} = \frac{e^{ia\eta}}{K_+(\eta)} - \frac{i}{2}\sum_{k=1}^{N} b_k^{\pm}$$

$$P_k^{\pm}(t) = b_k^{\pm}(1 - \operatorname{erf}\sqrt{-i(\varkappa + z_k)t})$$

$$S_k^{\pm}(t) = b_k^{\pm}\operatorname{erf}\sqrt{-i(\varkappa + z_k)t} \tag{3.16}$$

$$b_k^{\pm} = r_k\sum_{l=1}^{N}[B_{kl}^{+}\alpha(-z_l, +) \mp B_{kl}^{-}\alpha(-z_l, -)]$$

From (3.13) and (3.14), it is obvious that the initial stresses affect the singularity, which is oscillatory in character. If the multiplier $\exp(-i\omega t)$ is taken into account (a steady process is considered), it is clear that rapidly decaying waves move under the punch away from its edges, at a velocity equal to that of shear waves in the pre-stressed medium, governed by the initial stresses. Similarly, on the free surface [expressions (3.15)], there are rapidly decaying waves, the velocity of which is also that of shear waves.

4. THE CASE OF A PLANE PUNCH

By way of example, we will consider the problem of shear vibrations of a plane punch [putting $\eta = 0$ in (3.10)]. From (3.12)–(3.15) it follows that

$$q(x) = q_0(x) + q_1(x) + q_2(x), \quad |x| \leqslant 1$$

$$\varphi^{\pm}(x) = R(x \mp 1), \quad \pm x > 1$$

$$q_0(x) = -i\varkappa[-1 + L(1 - x) + L(1 + x)] \tag{4.1}$$

$$q_m(x) = M_m(1 - x) + M_m(1 + x), \quad m = 1, 2$$

$$R(t) = 1 - \operatorname{erf}\sqrt{-i\varkappa t} + \frac{i}{2}\sum_{k=1}^{N} K_-(z_k)\, S_k^{+}(t)\, e^{iz_k t}$$

$$L(t) = \operatorname{erf}\sqrt{-i\varkappa t} + \frac{1}{\sqrt{-i\varkappa}}\frac{e^{i\varkappa t}}{\sqrt{\pi t}}$$

$$M_1(t) = -\frac{i}{2}\sum_{k=1}^{N} b_k\frac{e^{i\varkappa t}}{\sqrt{\pi t}}$$

$$M_2(t) = \frac{i}{2}\sum_{k=1}^{N} b_k\sqrt{-i(z_k - \varkappa)}\, e^{-iz_k t}(1 - \operatorname{erf}\sqrt{-i(\varkappa + z_k)t})$$

$$b_k = r_k\sum_{l=1}^{N} B_{kl}^{+}(-z_l, +)$$

Formulas (4.1) give quite a clear idea of the structure of the wave field both under the punch and on the free surface.

The amplitudinal value of the reactive force acting on the punch from the half-space side has the form

$$\mathbf{P} = \int_{-1}^{1} \mathbf{q}(\mathbf{x})\,d\mathbf{x} = \mathbf{P}_0 + \sum_{k=1}^{N} \frac{b_k}{\mathbf{z}_k}[-\sqrt{-i\varkappa}\,\mathrm{erf}\sqrt{-2i\varkappa} + $$
$$+\sqrt{-i(\mathbf{z}_k+\varkappa)}\,(1-e^{-2i\mathbf{z}_k}(1-\mathrm{erf}\sqrt{-2i(\varkappa+\mathbf{z}_k)}))] \qquad (4.2)$$
$$\mathbf{P}_0 = 2i\varkappa + (1-4i\varkappa)\,\mathrm{erf}\sqrt{-2i\varkappa} + 2\sqrt{-2i\varkappa/\pi}\,\exp(2i\varkappa)$$

The parameter $\varkappa$ is related to the initial deformation ($\varkappa = \omega a(\rho/A_2)^{1/2}$), and therefore the initial stresses have a direct influence both on the character of the wave field and on the resultant of the contact stresses. Knowing, for instance, the behaviour of the coefficient A_2, with the help of (4.1) and (4.2) the features of the interaction of the punch with the pre-stressed medium are fairly easy to analyse. In particular, as calculations have shown, for most materials [10] a positive initial deformation (tension) causes a reduction in A_2 while a negative initial deformation results in an increase in A_2. Thus, initial compression of the medium leads to an increase, tension leads to a decrease in vibrations. We note that, in the absence of initial stresses ($A_2 = 1$), the degenerate component of the contact stresses $\mathbf{q}_0(x)$, as well as the component $\mathbf{P}_0$ of the reaction of the medium, are identical with the asymptotic solution obtained for large frequencies of vibration in [12].

5. NUMERICAL ANALYSIS

Formulas (4.1) and (4.2) were constructed without reference to the form of the initial stressed state, or to the properties or form of the law of state of the material of the medium. Before continuing the investigation, it is necessary to specify the properties of the medium. For that purpose, we shall assume that the material is compressible and initially isotropic, with an elastic potential. For the latter we can use, for example, the potential in the Murnaghan form [6, 7]

$$W^\circ = {}^1/_2\lambda B_1^{\circ 2} + \mu B_2^\circ + {}^1/_3 aB_1^{\circ 3} + bB_1^\circ B_2^\circ + {}^1/_3 cB_3^\circ \qquad (5.1)$$

where λ and μ are Lamé constants and a, b and c are third-order constants. Assuming that the initial deformation is uniform (λ_i = const), from (2.2), using (5.1), we have

$$\mu_{ik} = \mu + bB_1^\circ + {}^1/_4 c(\mathbf{r}_i + \mathbf{r}_k)$$
$$\sigma_{ii}^{*\circ} = \lambda B_1^\circ + aB_1^{\circ 2} + bB_2^\circ + \mathbf{r}_i(\mu + bB_1^\circ) + {}^1/_4 c\mathbf{r}_i^2 \qquad (5.2)$$
$$B_1^\circ = {}^1/_2\sum_{n=1}^{3}\mathbf{r}_n, \quad B_2^\circ = {}^1/_4\sum_{n=1}^{3}\mathbf{r}_n^2, \quad B_3^\circ = {}^1/_8\sum_{n=1}^{3}\mathbf{r}_n^3, \quad \mathbf{r}_n = \lambda_n^2 - 1$$

We will assume that the initial stressed state for this problem is defined by the condition

$$\sigma_{11}^{*\circ} \neq \sigma_{22}^{*\circ} = \sigma_{33}^{*\circ} = 0 \qquad (5.3)$$

A numerical analysis shows that the distribution of the contact stresses and the behaviour of the free surface, calculated with (4.1) and (4.2) when there is no initial deformation, are of the usual type.

The material used for the numerical calculations (35KhGSA steel [7, 10]) had the following parameters: $\rho = 7.748\times 10^3$ kg/m^3, $\lambda = 1.1\times 10^{11}$ N/m^2, $\mu = 0.804\times 10^{11}$ N/m^2, $a = -7.09\times 10^{11}$ N/m^2, $b = 0.77\times 10^{11}$ N/m^2, $c = -8.04\times 10^{11}$ N/m^2.

The graphs of the functions [$\mathbf{q}^\circ(\mathbf{x}) = \mathbf{q}(\mathbf{x})$ when there is no initial deformation in the medium], Re$\mathbf{q}^\circ(\mathbf{x})$ (solid lines) and Im$\mathbf{q}^\circ(\mathbf{x})$ (dashed lines) are shown in Fig. 1 for different frequencies of vibration of the punch (curves 1–3 correspond to parameter values $\varkappa = 0.03$, 0.3 and 1.55). It is clear that at low frequency ($\varkappa = 0.03$), the distribution of contact stresses is similar to the static distribution [Re$\mathbf{q}^\circ(\mathbf{x})$ remains of the same sign and is considerably larger than Im$\mathbf{q}^\circ(\mathbf{x})$]. At a medium frequency ($\varkappa = 0.3$) the solution is of the same character but Im$\mathbf{q}^\circ(\mathbf{x})$ increases rapidly and becomes larger than Re$\mathbf{q}^\circ(\mathbf{x})$. At a high frequency ($\varkappa = 1.55$), the stress distribution under the punch becomes oscillatory, owing to the fact that the wavelength of the shear wave excited by the edges of the punch becomes smaller than the size of the punch. The addition of oscillatory terms to the penetrating component $\mathbf{q}_0(\mathbf{x})$ (4.1) transforms Im$\mathbf{q}^\circ(\mathbf{x})$ to saddle-shaped form.

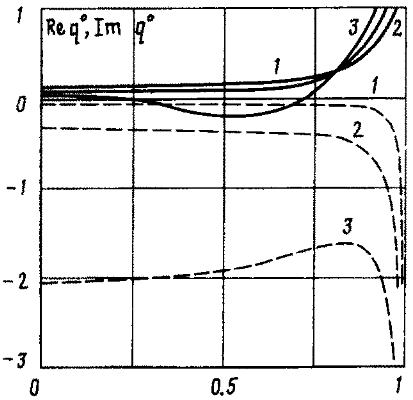

FIG. 1.

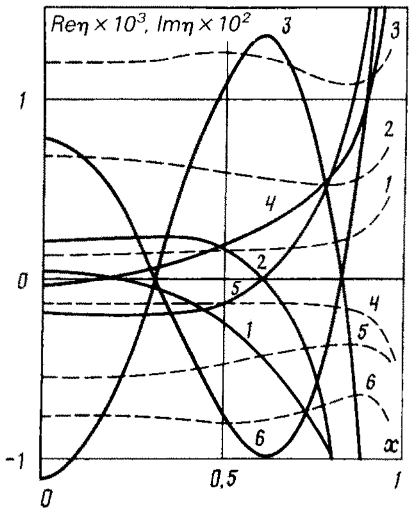

FIG. 2.

Figure 2 shows graphs of the functions [$\eta(\mathbf{x}) = \mathbf{q}^\circ(\mathbf{x}) - \mathbf{q}(\mathbf{x})$ is the change of the contact stresses] $\mathrm{Re}\,\eta(\mathbf{x})$ (solid curves) and $\mathrm{Im}\,\eta(\mathbf{x})$ (dashed curves) as a function of the initial deformation and frequency [curves 1–3 correspond to $\lambda_1 = 1.005$ (tension) for values of $\varkappa = 0.03$, 0.3 and 1.55 and curves 4–6 to $\lambda_1 = 0.995$ (compression) at the same frequencies]. The graphs give a clear illustration of the influence of initial deformation on the distribution of contact stresses under the punch at low, medium and high frequencies. The points at which $\mathrm{Re}\,\mathbf{q}^\circ(\mathbf{x})$ is independent of the initial deformation (the intersection of the solid curves 1 and 4, 2 and 5, 3 and 6) are of special interest. Their position and number obviously depend very much on the frequency and are due to the presence of the oscillating components in $\mathbf{q}(\mathbf{x})$ (4.1). The difference in behaviour of $\mathrm{Re}\,\eta(\mathbf{x})$ and $\mathrm{Im}\,\eta(\mathbf{x})$ is due to the predominant influence of the initial deformation on the penetrating component (which is constant at fixed frequency) of $\mathbf{q}(\mathbf{x})$ (4.1).

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